Construction of p-adic L-functions for unitary groups II

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Joint work w/ Michael Harris & J.S. Li

To construct p-adic L-fns for (families of) automorphic forms on unitary gps of arb. signature over a totally real fld.

<u>This talk:</u> explain this construction in a simple setting

- work with an im. quad fld (instead of general CM flds)
- stick to definite unitary gps
- work with simple automorphy types ← a more serious limitation (need a theory of p-adic differential operators o/w.)

<u>Situation</u>: F: imag. quad fld $F \subset \bar{\mathbb{Q}} \subset \mathbb{C} \simeq \mathbb{C}_p$.

p prime that splits in F. picks out a place above p, $v|p$ in F.

$\rho: G_F \to GL_n(\bar{\mathbb{Q}}_p)$ cont. geometric

(PC) $w|p$ $\rho|_{D_w} \simeq \begin{pmatrix} \sigma_w^- & * \\ & \sigma_w^+ \end{pmatrix}$ (2-step filtration)

σ_w^- HT wts all <0 σ_w^+ HT wts ≥ 0.

$\dim \sigma_w^- = \dim \sigma_{\bar{w}}^+$ ← the conjugate wt (under cx conj.)

<u>Expectation</u>: (Coates, Perrin-Riou, ...)

The values $L(\rho \otimes \chi, 0)$ are interpolated by a p-adic L-fcn.

e.g. F_∞/F max'l $\mathbb{Z}_p$-extension, $\Gamma_F := Gal(F_\infty/F) \simeq \mathbb{Z}_p^2$

χ finite char. of Γ_F.

More precisely, there should exist $\mathcal{L} = \mathcal{L}_p \in \mathcal{O}[[\Gamma_F]]$.

s.t.

$$\mathcal{L}(\chi) = \underset{\substack{\text{periods} \\ \text{powers of } p}}{c(*)} \; \underline{\text{additional factor}} \; L^p(\rho\otimes\chi, 0)$$

additional factor $= \prod_{w|p} \frac{L(0, \sigma_w^- \chi_w)}{L(1, (\sigma_w^- \chi_w)^\vee)}$

Examples:

① $n=1$

ψ: Hecke character of $\mathbb{A}_F^\times$, $\psi_\infty(z)=z^k$, $k>0$

$\sigma_\psi : G_F \to \overline{\mathbb{Q}}_p^\times$, $L(\sigma_\psi, s) = L(s,\psi)$

v : HT $-k$ expect a p-adic L-fcn interpolating

$\bar v$: HT 0

(*) $L^{p}(0,\psi\chi)(1-\psi_v\chi_v(\varpi_v))^{-1}(1-\psi_v^{-1}\chi_v^{-1}(\varpi_v)p^{-1})$

This was constructed by Katz

② f wt 2 eigenform

ρ_f - usual p-adic rep'n assoc. to f

ψ as in ①

(fn PC)

ⓐ $k=0$ $\rho=\rho_f\otimes\sigma_\psi\varepsilon$ $L(f,\psi,1)$ ← f ordinary at p.

ⓑ $k\geq 2$ $\rho=\rho_f\otimes\sigma_\psi$ $L(f,\psi,0)$

p-adic L-fcn constructed by Hida, Perrin-Riou, ...

③ W: definite Hermitian space over F of dim n

e.g. $\langle x,y\rangle = x\cdot{}^t\bar y$

$G=U(W)$ unitary gp over $\mathbb{Q}$

π: cuspidal autom. rep'n of $G(\mathbb{A})$ with trivial minimal K_∞-type

One expects (often knows, e.g. if π is square-integrable at one ∞-place or two finite places)

that $\exists$ $\rho_\pi : G_F \to GL_n(\overline{\mathbb{Q}}_p)$ s.t.

- HT wts at v, $\bar v$ $0,1,\dots,n-1$
- $\rho_\pi\circ c \simeq \rho_\pi^\vee \otimes \varepsilon^{n-1}$
- $L(\rho_\pi,s)\equiv L(\pi, s-\frac{n-1}{2}, St)$ (base changed)

ψ as in ① with $k\geq n$

dim of negative space w/ HT wts $\rho=\rho_\pi\otimes\sigma_\psi$

↙ but add in the factors

dim [illegible] neg space w/ HT wts

$\rho = \rho_\pi \otimes \sigma_{\psi}$

$$\boxed{\begin{array}{c} \dim \sigma_v^- = n \\ \parallel \\ \dim \sigma_{\bar v}^+ = n \end{array}}$$

P.C.

HT v: $-k, 1-k, \ldots, n-1-k$ < 0 ← but add in the factors here

$\bar v$: $0, \ldots, n-1 \geq 0$ ← remove factors here

p-adic L-function should interpolate:

$$L^p(0, \rho_\pi \otimes \psi_x) \cdot \frac{L_v(0, \rho_\pi \otimes \psi_x)}{L_v(1, (\rho_\pi \otimes \psi_x)^\vee)} \quad (*).$$

$$L^p(0, \rho_\pi \otimes \psi_x) = L^p\left(\pi, \psi' x, \tfrac{k-n}{2}\right)$$

This construction subsumes examples ① and ②b.

Key ingredient in the construction of the p-adic L-function is the doubling method

$2W = W \oplus (-W)$ signature (n,n) (under all possible embeddings into $\mathbb{C}$)

$H = \mathcal{U}(2W)$ $(\mathcal{U}(n,n))$

$W^d \subset 2W$ maximal isotropic sub-space.

$P \subseteq H$
$= \mathrm{Stab}(W^d)$

$P = MN$ $M \simeq GL(W^d)$

χ - unitary Hecke character of $\mathbb{A}_F^\times$, $\chi_\infty = z^k |z|^{-k}$

$f \in \mathrm{Ind}_P^H(\chi \cdot \delta^s)$
$= f(h,s)$ $\delta(g) = |\det m|_F$ $g = mnk \in MNK_{\mathbb{A}}$. K a fixed maximal compact.

One constructs

$$E_f(h,s) = E_f(h,\chi,s)$$
$$= \sum_{\gamma \in P(\mathbb{Q})\backslash H(\mathbb{Q})} f(\gamma h)$$ meromorphic continuation by Langlands and others

Michael's talk is on choosing f so E_f behaves well.

π : representation of $G(\mathbb{A})$ $(\mathcal{U}(W))$ Note $G \times G \hookrightarrow H$,

$\varphi \in \pi$, $\varphi' \in \pi^\vee$, $\varphi'_\chi = \varphi' \otimes \chi^{-1}$ "$\varphi'(g) \cdot \chi^{-1}(\det g)$".

$$Z(s,\varphi,\varphi',f,\chi) = \int_{(G\times G)(\mathbb{Q})\backslash (G\times G)(\mathbb{A})} E_f((g,g')) \varphi(g) \varphi'_\chi(g')\, dg\, dg'$$

$\varphi = \otimes_v \varphi_v$ $\varphi' = \otimes \varphi'_v$ $f = \otimes f_v$

$Z(s) = \prod Z(s)$

$$Z(s) = \prod \underbrace{Z_v(s)}_{\displaystyle \overset{\shortparallel}{\int_{G(\mathbb{Q}_v)} f_v((g,1)) \langle \pi(g)\varphi, \varphi' \rangle_v \, dg}}$$

everything unramified

$$Z_v(s) = L_v(\pi, \chi, s+\tfrac{1}{2}, St) \langle \varphi_v, \varphi_v' \rangle_v$$

<u>problem</u>: dealing with the ramified places.

$S \supseteq \{\infty, p,$ ℓ at which π, χ, or G is ramified $\ell \mid \mathrm{disc}\, F\}$.

$\underline{\ell \notin S}$: unramified data

$\underline{\ell \neq p, \infty}$: can choose f_ℓ to have small support $Z_v(s) = \text{constant}$

$\underline{\ell = \infty}$: for simple minimal K_∞-types, can choose a general f_∞ (Garrett, Harris, J.-S. Li)

$\underline{\ell = p}$: Michael explained a good choice of f_p

"good" local Fourier coeff. has a simple form, interpolated p-adically by inspection.

$\underline{1^{st}\text{ talk}}$

$\underline{2^{nd}\text{ talk}}$ the local zeta integral $Z_v(s)$ using this section has the correct shape

computation carried by J.-S. Li.

$$\Phi \in \mathcal{S}(2W_p) \qquad \int_{GL(W_v^d)} \Phi((z,z)h) \cdots dz \quad \cdots$$

How to then make a p-adic L-fcn ?

<u>strategy for constructing p-adic L-fcn.</u>

"Katz"

$GL_2 \leftrightarrow U(1,1)$
$U(1)$

- goes back to Katz.

<u>Geometric part</u>: ψ', $\psi'_\infty = z^k|z|$, $\psi = \psi' \cdot | \; |_F^{-k/2}$

① Eisenstein measure

measure on $T_F = 1 + p\mathcal{O}_{F,p}$ taking values in <u>space of</u> <u>p-adic modular forms</u> for H
(Hida)

$$\chi \longmapsto \mathbb{E}(\chi) = E_f(h, \psi'\chi, \tfrac{k-n}{2})$$

② restrict to p-adic modular forms on $G \times G$

③ pair against forms on $G \times G$ related to π.

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Automorphic part:

reinterpret the pairing as the Zeta integral

$\mathcal{U}(n) \times \mathcal{U}(n)$

$\downarrow$ understood.

$\mathcal{U}(n,n)$

· periods show up.

Once this is carried out, one gets the p-adic L-fcn $\mathcal{L}_{p,\pi\otimes\sigma_{\mathbb{Z}_p}}$

The next step is to vary these (p-adic) L-fcns in families

Can allow π to vary (as long as (PC) is satisfied)

e.g. Hida family

$\mathrm{sig}(W) = (\dim \sigma_v^+, \dim \sigma_v^-)$

W - depends herm. space

$L_p \subset W_p$ self-dual lattice (assumption)

$\|$

$L_v \oplus L_{\bar{v}}$ $\qquad$ L_v defines a model of $G/\mathbb{Z}_p$.

$$G/\mathbb{Q}_p \subseteq GL(W_p) = GL(W_v) \times GL(W_{\bar{v}})$$

$$G/\mathbb{Q}_p \xrightarrow{\sim} GL(W_v)$$

$$G/\mathbb{Z} \longleftrightarrow GL(L_v)/\mathbb{Z}_p = G_v.$$

$GL_n/\mathbb{Z}_p$ (fixing basis)

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standard $B \subset GL(L_0) = G/\mathbb{Z}_p$

standard $Q \supset B \leftrightarrow n = n_1 + \dots + n_\ell$

$$Q = \begin{pmatrix} \boxed{n_1 \times n_1} & & * \\ & \boxed{n_2 \times n_2} & \\ & & \ddots \\ & & & \boxed{n_\ell \times n_\ell} \end{pmatrix}$$

$= M_Q N_Q$ $\quad g_1, \dots, g_\ell \longmapsto (\det g_1, \dots, \det g_\ell)$

$$M_Q \simeq GL_{n_1} \times \dots \times GL_{n_\ell} \longrightarrow \underbrace{\mathbb{Z}_p^\times \times \dots \times \mathbb{Z}_p^\times}_{\ell}$$

Automorphic forms on G

χ - alg char. of $GL_n/\mathbb{Z}$. (e.g. $\det^a$)

$$G(\mathbb{R}) \subset G(\mathbb{C}) = G(\mathbb{C}_p) \supset G(\mathbb{Q}_p)$$

$G(\mathbb{Q})$

$K \subseteq G(\mathbb{A}_f)$ compact open $\quad \nu : K \longrightarrow \mathbb{C}^\times$ character.

$A(\chi, K, \nu)$

$$= \left\{ f : G(\mathbb{A}) \to \mathbb{C} \quad \text{s.t.} \quad f(\gamma g k_\infty k) = \chi(k_\infty)\, \nu(k) f(g) \atop \gamma \in G(\mathbb{Q}) \quad k_\infty \in G(\mathbb{R}),\ k \in K \right\}$$

Shift action to p

$$A(\chi, K, \nu) \xrightarrow{\sim} A_p(\chi, K, \nu)$$

$$= \left\{ f : G(\mathbb{A}_f) \to \mathbb{C}_p \atop f(\gamma g k) = \chi(k_p)\, \nu(k) f(g), \quad \gamma \in G(\mathbb{Q}),\ k \in K \right\}$$

$k_p k^{(p)}$?

$$f \longmapsto (g \mapsto \chi(g_p) f(g))$$

$K = G(\mathbb{Z}_p) K^p$

$K_Q(m) = \{ k \in K \mid k_p \bmod p^m \in Q'(\mathbb{Z}/p^m) \}$

$$A_Q^{p\text{-adic}}(K^p, R) := \varprojlim_r \varinjlim_m A_p(K_Q(m), R/p^r)$$ action of $M_Q(\mathbb{Z}_p)$

$= GL_{n_1}(\mathbb{Z}_p) \times \dots \times GL_{n_\ell}(\mathbb{Z}_p)$

R = p-adic ring

$A_Q^{p\text{-adic}}[\underline{\nu}]$

acting through

$M_Q(\mathbb{Z}_p) \to \mathbb{Z}_p^\times \times \dots \times \mathbb{Z}_p^\times$

Can define measure ....

computation seems involved ...

$M_{\mathbb{Q}}(\mathbb{Z}_p) \to \underbrace{\mathbb{Z}_p^{\times} \times \cdots \times \mathbb{Z}_p}_{l \text{ times.}}$

$\underline{v} = (v_1, \ldots, v_l)$